

ISOMORPHISMS PRESERVING INVARIANTS

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ABSTRACT. Let V and W be finite dimensional real vector spaces and let $G \subset \mathrm{GL}(V)$ and $H \subset \mathrm{GL}(W)$ be finite subgroups. Assume for simplicity that the actions contain no reflections. Let Y and Z denote the real algebraic varieties corresponding to $\mathbb{R}[V]^G$ and $\mathbb{R}[W]^H$, respectively. If V and W are quasi-isomorphic, i.e., if there is a linear isomorphism $L: V \rightarrow W$ such that L sends G -orbits to H -orbits and L^{-1} sends H -orbits to G -orbits, then L induces an isomorphism of Y and Z . Conversely, suppose that $f: Y \rightarrow Z$ is a germ of a diffeomorphism sending the origin of Y to the origin of Z . Then we show that V and W are quasi-isomorphic. This result is closely related to a theorem of Strub [Str82], for which we give a new proof. We also give a new proof of a result of [KLM03] on lifting of biholomorphisms of quotient spaces.

1. INTRODUCTION

First some motivation. Let G be a finite group and let M and N be smooth G -manifolds. We give the orbit spaces M/G and N/G a smooth structure by declaring that $C^\infty(M/G) = C^\infty(M)^G$ and $C^\infty(N/G) = C^\infty(N)^G$. We also have a stratification of M/G by isotropy type, where the type of a point Gm of M/G is the conjugacy class of the isotropy group G_m of $m \in M$. Suppose that M and N are equivariantly diffeomorphic. Then M/G and N/G are “diffeomorphic” and the diffeomorphism preserves isotropy type. Conversely, suppose that M/G and N/G are diffeomorphic by a strata preserving diffeomorphism f . Can we lift f to an equivariant diffeomorphism of M and N ? First of all, f must give a diffeomorphism of M^G and N^G (assuming these sets are nonempty). Let $v \in M^G$ and set $V = T_v M$. Let $f(v) = w \in N^G$ and set $W = T_w N$. Then the differentiable slice theorem says that there is a neighborhood of v in M/G diffeomorphic to $U \times V/G$ where U is a neighborhood of v in M^G . It follows that f induces a strata preserving diffeomorphism of V/G with W/G . The first question to answer is if the existence of a diffeomorphism of V/G with W/G implies that V and W are isomorphic representations of G . The answer is “yes,” up to an automorphism of G , by a theorem of Strub [Str82]. We generalize the problem by replacing the requirement that strata are preserved by a weaker condition, we allow for a different group H to act on W and we allow the diffeomorphism f to be defined on the Zariski closures of the orbit spaces, in a sense made precise below. This ends our motivation, and the precise description of our results follows.

Let V and W be finite dimensional real vector spaces and let $G \subset \mathrm{GL}(V)$ and $H \subset \mathrm{GL}(W)$ be finite subgroups. Let Y and Z denote the real algebraic varieties corresponding to $\mathbb{R}[V]^G$ and $\mathbb{R}[W]^H$, respectively. We say that a linear isomorphism $L: V \rightarrow W$ is a *quasi-isomorphism* if L sends G -orbits to H -orbits and L^{-1} sends H -orbits to G -orbits. We say that V and W are *quasi-isomorphic* if such an L exists. If L is a quasi-isomorphism, then $g \mapsto L \circ g \circ L^{-1}$ gives an isomorphism of G and H such that, with G acting on W via the isomorphism, L is equivariant.

If V and W are quasi-isomorphic, then Y and Z are isomorphic. Conversely, we show that if $f: Y \rightarrow Z$ is a germ of a diffeomorphism sending the origin of Y to the origin of Z such that f maps the closures of the codimension one strata of Y onto those of Z , then V and W are quasi-isomorphic. The origin of Y is the image of $0 \in V$ in Y and similarly for Z .

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There are closely related results in the literature. Let Y_0 and Z_0 denote the images of V and W in Y and Z , respectively. Then Y_0 is isomorphic to the orbit space V/G and Z_0 is isomorphic to W/H . Strub [Str82] has shown that if there is a diffeomorphism of neighborhoods of the origins in Y_0 and Z_0 , then the G and H actions are quasi-isomorphic. Prill [Pri67] showed the analogue of Strub's result in the case of complex representations U of finite complex groups K not containing pseudoreflections. Here one considers the complex variety U/K corresponding to $\mathbb{C}[U]^K$. Gottschling [Got69] and Prill [Pri67] proved that biholomorphisms of U/K lift to equivariant biholomorphisms of U , again assuming that K contains no pseudoreflections. In [Los01] and [KLM03] there are results about lifting of isomorphisms of complex quotients $U/G \simeq U'/H$ where U and U' are complex G and H -modules, respectively. Lyashko [Lya83] showed that lifting holds in case $G = H$ is a Weyl group and $U = U'$ is the complexification of the standard real representation. Losik [Los01] proves that diffeomorphisms of Y_0 lift to diffeomorphisms of V which preserve G -orbits. We give new proofs of the lifting result of [KLM03] and of the main result of [Str82].

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2. ISOMORPHISMS OF SPACES OF INVARIANTS

Let p_1, \dots, p_m be homogeneous generators of $\mathbb{R}[V]^G$ and let q_1, \dots, q_n be homogeneous generators of $\mathbb{R}[W]^H$. Let d_j denote the degree of p_j , $j = 1, \dots, m$ and let e_j denote the degree of q_j , $j = 1, \dots, n$. Let $p = (p_1, \dots, p_m): V \rightarrow \mathbb{R}^m$ and $q = (q_1, \dots, q_n): W \rightarrow \mathbb{R}^n$. Let Y_0 denote $p(V)$ and let Z_0 denote $q(W)$. Then Y_0 and Z_0 are closed and semialgebraic where $Y_0 \simeq V/G$ and $Z_0 \simeq W/H$. We may identify Y and Z with the Zariski closures of Y_0 in \mathbb{R}^m and of Z_0 in \mathbb{R}^n . Since the p_i are homogeneous, Y_0 is stable under the \mathbb{R}^* action on \mathbb{R}^m which sends $t \in \mathbb{R}^*$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ to $t \cdot y := (t^{d_1} y_1, \dots, t^{d_m} y_m)$. It follows that Y is stable under this action. Similarly we have an action of \mathbb{R}^* on \mathbb{R}^n which preserves Z_0 and Z . Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of V . Then $p: V_{\mathbb{C}} \rightarrow \mathbb{C}^m$ has image the complex variety $Y_{\mathbb{C}}$ corresponding to $\mathbb{C}[V_{\mathbb{C}}]^G$. Moreover, Y is $Y_{\mathbb{C}} \cap \mathbb{R}^m$. The strata of Y and Y_0 are the collections of points whose preimages in $V_{\mathbb{C}}$ have conjugate isotropy groups. There are clearly finitely many strata. We say that a stratum is *codimension one* if it is of codimension one in Y_0 or Y , as the case may be. Equivalently, the isotropy groups corresponding to the stratum have order 2 and are generated by reflections. The *principal points* of Y and $Y_{\mathbb{C}}$ are those points whose inverse images in $V_{\mathbb{C}}$ have trivial isotropy groups.

Example 2.1. Let $V = \mathbb{R}$ and $G = \{\pm 1\}$ acting by multiplication. Then $p = x^2: \mathbb{R} \rightarrow \mathbb{R}$. We have that $Y_0 = \mathbb{R}^+$ and $Y = \mathbb{R}$. The codimension one stratum of Y is the origin and the principal points are $Y \setminus \{0\}$.

Example 2.2. Let $V = \mathbb{R}^2$ and $G = \mathbb{Z}/k\mathbb{Z}$ acting via rotations, $k \geq 2$. Let x and y be the usual coordinate functions on V and set $z = x + iy$. Then the G -invariant polynomials are generated by $z\bar{z}$ and the real and imaginary parts of z^k . Thus $p_1 = x^2 + y^2$, $p_2 = x^k - \binom{k}{2}x^{k-2}y^2 + \dots$ and $p_3 = kx^{k-1}y - \binom{k}{3}x^{k-3}y^3 + \dots$ generate $\mathbb{R}[V]^G$. They satisfy the relation $p_2^2 + p_3^2 = p_1^k$. Hence $Y = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^k = y_2^2 + y_3^2\}$ and one can show that $Y_0 = \{(y_1, y_2, y_3) \in Y \mid y_1 \geq 0\}$. If k is odd, then $Y = Y_0$, while Y and Y_0 differ if k is even. Here the strata are again $\{0\}$ and $Y \setminus \{0\}$, so there are no codimension one strata.

Assume that we have a germ of a diffeomorphism $f: Y \rightarrow Z$ sending 0 to 0. This means that f extends to a smooth germ of a mapping from \mathbb{R}^m to \mathbb{R}^n sending Y to Z and similarly for f^{-1} . We assume that f sends the closures of the codimension one strata of Y onto the closures of the codimension one strata of Z .

Lemma 2.3. *Let $f: Y \rightarrow Z$ be a local diffeomorphism as above. Then f maps principal points of Y to principal points of Z .*

Proof. Everything we do should be understood to happen in neighborhoods of 0 in Y and Z where f and f^{-1} are defined. Let C and D be the closures of the codimension one strata in Y and Z , respectively. It follows from the Shepherd-Todd-Chevalley theorem that $y \in Y$ is a smooth point if and only if the isotropy group of a preimage $y_0 \in V_{\mathbb{C}}$ is trivial or is generated by reflections. In the latter case let r be one of the reflections in G_{y_0} . Then y lies in the closure of the codimension one stratum consisting of points whose isotropy group is generated by r . Thus a point in $Y \setminus C$ is smooth in Y if and only if it is principal. Hence the principal stratum of Y consists of the smooth points not in C , so that f induces an isomorphism of the principal points of Y and Z . \square

Let $y \in Y$ be close to zero so that f is defined on $(0, 1] \cdot y$. For $t \in (0, 1]$, let f_t denote $t^{-1} \cdot f(t \cdot y)$. We say that f is *quasilinear* if $f_t = f$ for all $t \in (0, 1]$. In this case, of course, f is the restriction of a global diffeomorphism of Y and Z which is induced by a polynomial mapping.

Lemma 2.4. *Let $f: Y \rightarrow Z$ be a local diffeomorphism as above. Then $f_t(y)$ converges uniformly to a limit $f_0(y)$ as $t \rightarrow 0$ for y in a neighborhood of 0 in Y . The mapping f_0 is a quasilinear polynomial isomorphism of Y and Z which preserves the closures of the codimension one strata.*

Proof. There are invariant inner products on V and W and we may assume that p_1 and q_1 are the corresponding quadratic forms. Let Y' and Z' denote the principal points of Y and Z , respectively. Then Y' and Z' are \mathbb{R}^* -stable and have finitely many components. The image $Z_1 \subset Z$ of the principal points of W is connected and open in Z and it is \mathbb{R}^* -stable. Since $q(W) = Z_0$ is closed in Z , any limit point of Z_1 which is not in Z_1 is not a principal point. Thus Z_1 is a connected component of Z' . It follows that $f^{-1}Z_1$ lies in and contains a neighborhood of 0 of a connected component Y_1 of Y' . Let $y \in Y_1$ such that f is defined on $(0, 1] \cdot y$. Set $c(t) = f(t \cdot y)$. Then $c(t) = (c_1(t), \dots, c_n(t))$ is a curve in Z_0 and from [Los01, Lemma 2.1] we see that $c_i(t)$ vanishes at least to order e_i at $t = 0$. Thus $f_t(y)$ converges to a limit $f_0(y)$. Now consider the Taylor polynomial Tf of f up to the maximum degree s of the q_i . Then the i th component $T_i(f)$ is a sum of monomials $\sum c_\alpha y^\alpha$ where $c_\alpha \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$. Let $|\alpha|$ denote $d_1\alpha_1 + \dots + d_m\alpha_m$. Then our calculation shows that the sum $\sum_{|\alpha| < e_i} c_\alpha y^\alpha$ vanishes on Y_1 . Since Y_1 is Zariski dense in Y the sum vanishes on Y . We may change f without changing its restriction to Y so that $T_i(f)$ has nonzero coefficients only for monomials y^α with $|\alpha| \geq e_i$. Then the Taylor polynomial of f up to degree s induces the mapping f_0 and the convergence of f_t to f_0 is uniform for $y \in Y$ in a neighborhood of 0. Since f_0 is a limit of maps preserving the closures of the codimension one strata, f_0 does also, and f_0 is clearly quasilinear. It is also invertible, with inverse $(f^{-1})_0$. \square

Remark 2.5. It is clear that f_0 induces a complex isomorphism of $Y_{\mathbb{C}}$ and $Z_{\mathbb{C}}$, also denoted f_0 , preserving the closures of the codimension one strata. This follows from the fact that the Zariski closures of the strata of Y_0 in $Y_{\mathbb{C}}$ are the strata of $Y_{\mathbb{C}}$, and this correspondence preserves codimension [Sch80, 5.8].

Remark 2.6. In Examples 2.1 and 2.2 (with $W = V$ and $H = G$) the mapping f_0 is linear. This is obvious in Example 2.1 since the generators are all quadratic. Example 2.2 requires more work.

Corollary 2.7. *Let $f_0: Y_{\mathbb{C}} \rightarrow Z_{\mathbb{C}}$ be a quasilinear isomorphism which induces an isomorphism of the closures of the codimension one strata. Then there is a quasi-isomorphism $F: V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ which induces f_0 .*

Proof. The proof of Lemma 2.3 shows that $f_0: Y_{\mathbb{C}} \rightarrow Z_{\mathbb{C}}$ induces an isomorphism of the principal points. Since it also preserves the closures of the codimension one strata, it has a biholomorphic lift $F: V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ such that F and F^{-1} preserve orbits [KLM03]. Let $F_t(v) = t^{-1}F(tv)$ for $t \in \mathbb{R}^*$ and $v \in V$. Then F_t is a continuous family of biholomorphic lifts of f_0 , so that we must have $F_t = F$ for all t and hence $F = \lim_{t \rightarrow 0} F_t$ is linear. Thus F is a quasi-isomorphism. \square

Let $h \in H$. Set $W_h := W_{h+} \oplus iW_{h-}$ where $W_{h\pm}$ denotes the ± 1 -eigenspace of h .

Lemma 2.8. *Let $z' \in Z$. Then there is an $h \in H$ such that $z' \in p(W_h)$.*

Proof. Let $z \in W_{\mathbb{C}}$ such that $q(z) = z'$. Since the polynomials making up q have real coefficients, $q(z) = \overline{q(\bar{z})} = q(\bar{z})$. Since $q(z) = q(\bar{z})$, there is an $h \in H$ such that $\bar{z} = hz$. Then $z = w_1 + iw_{-1}$ for some $w_1 \in W_{h+}$ and $w_{-1} \in W_{h-}$. \square

Theorem 2.9. *Let $f: Y \rightarrow Z$ be a local diffeomorphism sending 0 to 0 which preserves the closures of the codimension one strata. Then there is a quasi-isomorphism $F: V \rightarrow W$.*

Proof. We have shown that there is a quasi-isomorphism $F: V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ which induces f_0 on Y . Thus $F(V)$ has to be an H -stable linear subspace of $W_{\mathbb{C}}$ which maps to Z . Since H is finite, it follows from Lemma 2.8 that $F(V) = W_h$ for some $h \in H$ such that $h^2 = 1$ and such that h is central in H (else $F(V)$ is not H -stable). Since W_h is H -equivariantly isomorphic to W , we see that V and W are quasi-isomorphic. \square

Remark 2.10. Let F , h and f be as in the proof above. The linear mapping from $\mathbb{R}[W]^H$ to $\mathbb{R}[W]^H$ which sends $f(w_1 + w_{-1})$ to $f(w_1 + iw_{-1})$ induces a graded automorphism of $\mathbb{R}[W]^H$, hence a “linear” automorphism φ of Z . Then φ identifies Z_0 and $q(W_h)$. Our diffeomorphism f is just a local diffeomorphism of Y preserving Y_0 followed by the identification of pairs $Y_0 \subset Y$ with $q(W_h) \subset Z$ induced by the quasi-isomorphism F .

Our techniques establish the following theorem of Strub [Str82].

Theorem 2.11. *Let $f: Y_0 \rightarrow Z_0$ be a local diffeomorphism sending 0 to 0. Then there is a quasi-isomorphism $F: V \rightarrow W$.*

Proof. The principal points of Y_0 are exactly the points where it is locally a smooth manifold without boundary and the codimension one strata are exactly the boundary of the points where Y_0 is locally a manifold with boundary. Thus the analogues of Lemmas 2.3 and 2.4 hold for f and f_0 . Since f_0 must automatically give an isomorphism of Y and Z , the proof of Theorem 2.9 goes through. \square

Finally, we prove a result on lifting global isomorphisms generalizing [Los01, Theorem 3.4].

Theorem 2.12. *Let $f: Y_0 \rightarrow Z_0$ be a diffeomorphism. Then there is a lift $F: V \rightarrow W$ of f where F maps G -orbits to H -orbits.*

Proof. Let f_t be as in Lemma 2.4. Then there is a quasi-isomorphism $F': V \rightarrow W$ covering f_0 . Now $f_0^{-1}f_t$ is an isotopy of the identity of Y_0 . By the isotopy lifting theorem for finite groups [Bie75] or [Sch80], we may lift $f_0^{-1}f_t$ to an equivariant isotopy $F_t: V \times [0, 1] \rightarrow V$. Then $F' \circ F_1: V \rightarrow W$ is the desired lift of f . \square

3. LIFTING BIHOLOMORPHISMS

We have used lifting results from [KLM03]. The proofs there use results about connections or braid groups. Here we sketch how one can use the slice theorem for finite group actions to obtain their lifting result.

We change notation. Let $G \subset \mathrm{GL}(V)$ and $H \subset \mathrm{GL}(W)$ be finite subgroups where V and W are finite dimensional complex vector spaces. Let Y denote V/G (the variety associated to

$\mathbb{C}[V]^G$) and let Z denote W/H . If $g \in G$ is a pseudoreflection, let r_g denote its order and let $V^{<g>} \subset V$ denote the points of V^g whose isotropy group is generated by g . Then C , the union of the strata of codimension one, is a disjoint union $C_1 \cup \cdots \cup C_k$ where each C_j is the image of a subset $V^{<g_j>}$ for a pseudoreflection g_j , $j = 1, \dots, k$. To the component C_j we associate the number $r_j := r_{g_j}$. Similarly we have the union D of the codimension one strata of Z and a decomposition $D = D_1 \cup \cdots \cup D_l$ and pseudoreflections $h_1, \dots, h_l \in H$ with orders s_1, \dots, s_l such that D_j is the image of $W^{<h_j>}$, $j = 1, \dots, l$.

Theorem 3.1. *Suppose that $k = l$ and that $f: Y \rightarrow Z$ is a biholomorphism which sends each $\overline{C_i}$ onto a $\overline{D_j}$ such that $r_i = s_j$. Then there is a biholomorphic map $F: V \rightarrow W$ such that F and F^{-1} send orbits to orbits, and F induces f .*

Remark 3.2. It is easy to see that the conditions of the theorem are necessary for F to exist.

Proof of Theorem 3.1. We use a version of analytic continuation. Let V_1 denote the union of the reflection hyperplanes of V and let V_2 denote the union of the fixed subspaces V^K of subgroups K of G such that $\text{codim}_V V^K \geq 2$. Let Y_j denote the image of V_j in Y , $j = 1, 2$. Similarly we have subsets $W_j \subset W$ and $Z_j \subset Z$, $j = 1, 2$. Let $v \in V_1 \setminus V_2$, let g denote a reflection which generates the isotropy group of v and let $w \in W^h \setminus W_2$ be a point lying above $f(p(v))$ where h is a reflection generating the isotropy group of w . By hypothesis, g and h both have the same order, say r . By the slice theorem there is a ball B_1 about v in V^g and a ball B_2 in \mathbb{C} around 0 such that a G -neighborhood of v is isomorphic to $G \times^{(g)} (B_1 \times B_2)$ where (g) denotes the subgroup generated by g . Here g acts trivially on B_1 and acts on B_2 via multiplication by a primitive r th root of unity. We use the notation $[g_1, x, y]$ to denote the point in $G \times^{(g)} (B_1 \times B_2)$ corresponding to $g_1 \in G$, $x \in B_1$ and $y \in B_2$. Then the quotient mapping sends a point $[g_1, x, y]$ to (x, y^r) for $x \in B_1$ and $y \in B_2$. Similarly, there are balls B'_1 and B'_2 about w in W^h and $0 \in \mathbb{C}$ such that the quotient mapping sends $[h_1, x', y'] \in H \times^{(h)} (B'_1 \times B'_2)$ to $(x', (y')^r)$, $h_1 \in H$, $x' \in B'_1$, $y' \in B'_2$. The hypotheses on f show that for $(x, y) \in B_1 \times B_2^r$, $f(x, y) = (f_1(x, y), f_2(x, y)) \in B'_1 \times (B'_2)^r$ where f_2 vanishes when $y = 0$ and the derivative of f_2 in y has rank 1 along the zero set of y . It follows that, locally, $f_2(x, y)$ can be written as $m(x, y)y$ where $m(x, 0)$ does not vanish. Thus $f_2(x, y^r)$ has r holomorphic r th roots along $B_1 \times \{0\}$, so that we have r holomorphic lifts of f in a neighborhood of v which send v to w . The lifts are distinguished by their values at any point $(x, y) \in B_1 \times B_2$ where $y \neq 0$.

Now let $\gamma(t)$, $0 \leq t \leq 1$, be a continuous curve in $V \setminus V_2$ starting at a base point v_0 . Let F_0 be a germ of a holomorphic map from $V \rightarrow W$ which covers f . Since $W \setminus W_1 \rightarrow Z \setminus Z_1$ is a cover, we have an analytic continuation of F_0 to F_t at $\gamma(t)$ as long as $\gamma(t)$ never leaves $V \setminus V_1$. However, our argument above shows that we can continue F even if $\gamma(t)$ lands in $V_1 \setminus V_2$. Thus we may construct a continuous family F_t of lifts of f along $\gamma(t)$. The family is uniquely determined by F_0 . Now $V \setminus V_2$ is simply connected, so that F_1 only depends upon $\gamma(1)$. Thus we have a lift of f to $V \setminus V_2$. Since V_2 has codimension 2 in V , our lift extends to all of V .

We have shown that there is a global lift F of f , and similarly there is a global lift F^{-1} of f^{-1} . Now for any $g \in G$, $F \circ g \circ F^{-1}$ is an automorphism of W which covers the identity on W/H . Thus it must agree with an element of H . It follows that $g \mapsto F \circ g \circ F^{-1}$ gives an isomorphism of G and H , and that F and F^{-1} send orbits to orbits. \square

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